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EQUILIBRIUM STABILITY CONDITIONS UNDER 1:3 RESONANCE*

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The problem of asymptotic stability of the equilibrium position of an autonomous system of differential equations is examined. It is assumed that the linearized system's matrix has two pairs of pure imaginary eigenvalues and that the frequencies ratio equals three. Algebraic sufficient conditions for stability and instability are obtained.

1. Statement of the problem. We examine the differential equation system

$$du / dt = f(u),\ f(0) = 0,\ \dim u = \dim f = n \tag{1.1}$$

We study the asymptotic stability of the equilibrium position $u = 0$. Regarding the matrix $\Lambda = \| (df / du)_{u=0} \|$ we assume that: 1) there are two pairs of pure imaginary eigenvalues, i.e., $\lambda_{1,2} = \pm i\omega_1$ and $\lambda_{3,4} = \pm i\omega_2$, $\omega_2 \geqslant \omega_1 > 0$; 2) $\operatorname{Re} \lambda_j < 0$ for the remaining λ_j. The stability conditions for ω_k of general position were found in /1/. Several special cases exist in which the general criterion is inapplicable. These cases correspond to integral (resonance) relations between the frequencies.

$$\omega_2 = \omega_1\ (1:1);\quad \omega_2 = 2\omega_1\ (1:2);\quad \omega_2 = 3\omega_1\ (1:3)$$

In case of (1:2) the equilibrium position, as a rule, is unstable /2,3/. The same is true for the case of (1:1) if a Jordan cell corresponds to the eigenvalue $i\omega$ (**). If two eigenvectors correspond to eigenvalue $i\omega$, then the case (1:1) proves to be completely analogous to (1:3).

Below we consider the (1:3) case for which there is no criterion specified by explicit formulas: to investigate the stability of the steady-states solution it is necessary to make a detailed study of the phase portrait of an auxiliary system of two differential equations /4/. The paper's basic purpose is to derive simple sufficient stability and instability conditions. Examples are given in Section 8, showing that 1:3 resonance can lead to stability in those cases when there is no stability if it is not taken into account. We assume that the relation $\omega_2 = 3\omega_1$ is fulfilled exactly. The case when $\omega_2 \approx 3\omega_1$ was analyzed earlier (***). Below we examine a fourth-order system (1.1) for which all eigenvalues of matrix Λ are pure imaginary. By virtue of the reduction theorem /5/, this does not lessen the generality of the arguments.

2. Original equations. We reduce system (1.1) to normal form up to terms of third order, inclusive, by the change of variables $u \to x$, $x = (x_1, x_2, x_3, x_4) \in R^4$. In complex notation we obtain

$$\begin{gathered} dz_k / dt = h_k(z) + r_k(z);\ |r_k(z)| \leqslant C|z|^4;\ k = 1, 2 \\ z_1 = x_1 + ix_2,\ z_2 = x_3 + ix_4;\ |z|^2 = |z_1|^2 + |z_2|^2 \end{gathered} \tag{2.1}$$

The model system is written as

$$\begin{aligned} & dz_k / dt = h_k(z) \\ & h_1 = i\omega_1 z_1 + z_1(A_{11}|z_1|^2 + A_{12}|z_2|^2) + B_1(z_1^*)^2 z_2 \\ & h_2 = i\omega_2 z_2 + z_2(A_{21}|z_1|^2 + A_{22}|z_2|^2) + B_2 z_1^3;\quad \omega_2 = 3\omega \end{aligned} \tag{2.2}$$

Asymptotic stability of system (2.2) is equivalent to that of the homogeneous system resulting from (2.2) by discarding the linear terms. The linear terms $P_1(z)$ and the cubic terms $P_3(z)$ in the normalized equations commute (when Jordan cells are absent in Λ). Therefore, the systems $v^\cdot = P_1(v)$ and $w^\cdot = P_3(w)$ can be solved independently. If $v(t, \gamma)$ $(v(0, \gamma) = \gamma)$ and $w(t, \gamma)$ $(w(0, \gamma) = \gamma)$ are the general solutions of these systems, then $z(t, \gamma) = v[t, w(t, \gamma)] = w[t, v(t, \gamma)]$ is the general solution of system $z^\cdot = P_1(z) + P_3(z)$ (see /6/, for example). If all the eigenvalues of Λ are pure imaginary, then $|z(t, \gamma)| = |w(t, \gamma)|$. This equivalence follows from the formulas derived below, independently of the general considerations. If $B_1 = B_2 = 0$, then system (2.2) has the same form as the nonresonance system

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* Prikl.Matem.Mekhan.,44,No.2,229-237,1980

**) Khazin, L. G., On resonance instability of the equilibrium position under multiple frequencies. Preprint Inst. Prikl. Mat. Akad. Nauk SSSR, No.97, Moscow, 1975; Khazina, G. G. and Khazin, L. G., Nonexistence of an algebraic asymptotic stability criterion under 1:1 resonance. Preprint Inst. Prikl. Mat. Akad. Nauk SSSR, No.112, Moscow, 1977.

***) Shnol', E. E. and Khazin, L. G., Nonexistence of an algebraic asymptotic stability criterion under 1:3 resonance. Preprint Inst. Prikl. Mat. Akad. Nauk SSSR, No.45, Moscow, 1977; Khazin, L. G. and Shnol', E. E., Investigation of asymptotic stability of equilibrium under 1:3 resonance. Preprint Inst. Prikl. Mat. Akad. Nauk SSSR, No.67, Moscow, 1978.

$$dz_k/dt = i\omega_k z_k + z_k \sum_{j=1}^{2} A_{kj} |z_j|^2, \quad k = 1, 2 \tag{2.3}$$

Let us recall some well-known /1/ results on the stability of system (2.3).

Criterion A. For the asymptotic stability of system (2.3) it is necessary and sufficient to fulfil the following conditions: 1) $a_{11} < 0$; 2) $a_{22} < 0$; 3) $\Delta = a_{11}a_{22} - a_{12}a_{21} > 0$ when $a_{12} > 0$ and $a_{21} > 0$.

If system (2.3) is asymptotically stable, it admits of a Liapunov function of form $L = k\rho_1 + \rho_2, \rho_j = |z_j|^2$. In this connection

$$dL / dt < -C |z|^4 \quad (c > 0) \tag{2.4}$$

Function L is a Liapunov function even when higher terms are present in (2.3). A strict nonfulfillment of Criterion A, i.e., the fulfillment of at least one of the inequalities; 1) $a_{11} > 0$; 2) $a_{22} > 0$; 3) $\Delta < 0$ when $a_{12} > 0$ and $a_{21} > 0$, leads to the instability of system (2.3) independently of the higher terms of the Taylor expansion. Such a situation is called structurally-stable instability; system (2.3) has a growing solution in the form of an invariant ray

$$\rho_2(t) = p\rho_1(t); \quad d\rho_1 / dt = \alpha\rho_1^2 \ (\alpha > 0), \quad p = \text{const} \tag{2.5}$$

3. Transformed equations. After the change of variables $z_k = \rho_k^{1/2} e^{i\varphi_k}$, $\rho_k > 0$, $k = 1, 2$, from system (2.2) we obtain

$$\begin{aligned}
\rho_1^{\cdot} &= 2\rho_1 (a_{11}\rho_1 + a_{12}\rho_2) + 2b_1\rho_1^{3/2}\rho_2^{1/2} \cos(\psi - \psi_1) \\
\rho_2^{\cdot} &= 2\rho_2 (a_{21}\rho_1 + a_{22}\rho_2) + 2b_2\rho_1^{3/2}\rho_2^{1/2} \cos(\psi - \psi_2) \\
\psi^{\cdot} &= (\alpha_{21} - 3\alpha_{11})\rho_1 + (\alpha_{22} - 3\alpha_{12})\rho_2 - 3b_1\rho_1^{1/2}\rho_2^{1/2} \sin(\psi - \psi_1) - b_2\rho_1^{3/2}\rho_2^{-1/2} \sin(\psi - \psi_2) \\
A_{kl} &= a_{kl} + i\alpha_{kl}, \quad B_1 = b_1 e^{-i\psi_1}, \quad B_2 = b_2 e^{i\psi_2} \ (b_k \geqslant 0), \quad \psi = \varphi_2 - 3\varphi_1
\end{aligned} \tag{3.1}$$

We make use of the homogeneity of (3.1) with respect to ρ_1 and ρ_2. In the variables $\rho_1 = R\cos\theta$, $\rho_2 = R\sin\theta$, $d\tau = Rdt$ $(0 \leqslant R < \infty,\ 0 \leqslant \theta \leqslant \pi/2)$, from (3.1) we obtain

$$\begin{aligned}
&d\theta / d\tau = f(\theta, \psi) = f_1(\theta) + f_2(\theta, \psi), \quad d\psi / d\tau = g(\theta, \psi) = g_1(\theta) + g_2(\theta, \psi) \\
&f_1(\theta) = 2\cos\theta\sin\theta\,[(a_{21} - a_{11})\cos\theta + (a_{22} - a_{12})\sin\theta] \\
&f_2(\theta, \psi) = 2\cos^{3/2}\theta\sin^{1/2}\theta\,[b_2\cos\theta\cos(\psi - \psi_2) - b_1\sin\theta \times \cos(\psi - \psi_1)] \\
&g_1(\theta) = (\alpha_{21} - 3\alpha_{11})\cos\theta + (\alpha_{22} - 3\alpha_{12})\sin\theta \\
&g_2(\theta, \psi) = -3\ b_1\cos^{1/2}\theta\sin^{1/2}\theta\sin(\psi - \psi_1) - b_2\cos^{3/2}\theta\ \sin^{-1/2}\theta \times \sin(\psi - \psi_2)
\end{aligned} \tag{3.2}$$

$$\begin{aligned}
&dR / d\tau = R\Pi(\theta, \psi), \quad \Pi(\theta, \psi) = \Pi_1(\theta) + \Pi_2(\theta, \psi) \\
&\Pi_1(\theta) = 2\,(a_{11}\cos^3\theta + a_{12}\cos^2\theta\sin\theta + a_{21}\cos\theta\sin^2\theta + a_{22}\sin^3\theta) \\
&\Pi_2(\theta, \psi) = 2\cos^{3/2}\theta\sin^{1/2}\theta\,[b_1\cos\theta\cos(\psi - \psi_1) + b_2\sin\theta \times \cos(\psi - \psi_2)]
\end{aligned} \tag{3.3}$$

Finding $\theta(\tau)$ and $\psi(\tau)$ from (3.2), from (3.3) we obtain

$$R(\tau) = R(0)\exp\left[\int_0^\tau \Pi(\theta(s), \psi(s))\,ds\right] \tag{3.4}$$

Note 1°. The interval $0 < \tau < \infty$ corresponds to the interval $0 < t < \infty$ since

$$\int_0^\infty R dt = \infty$$

on the solution of (3.1).

To each solution of system (2.2) corresponds a solution of system (3.1). Conversely, to each solution of (3.1) corresponds a family of solutions of (2.2), each of which is distinguished by the choice of $\varphi_1(t_0)$ (or $\varphi_2(t_0)$). The last two assertions are valid for segments of solution $z(t)$, on which neither $z_1(t)$ nor $z_2(t)$ vanishes. When $z_1 = 0$ or $z_2 = 0$ the variable ψ loses meaning and it becomes necessary to observe the correspondence between the solutions in more detail. The plane $z_1 = 0$ is invariant for (2.2), i.e., either $z_1(t) \equiv 0$ or $z_1(t)$ does not vanish. The solutions of (2.2) with $z_1(t) \equiv 0$ satisfy the equation

$$z_2^{\cdot} = i\omega_2 z_2 + A_{22} z_2 |z_2|^2 \tag{3.5}$$

Having set $\rho_1 \equiv 0$ in (3.1), we obtain the equation

$$\rho_2^{\cdot} = 2a_{22}\rho_2^2 \tag{3.6}$$

The corresponding between the solutions of (3.5) and (3.6) is the same as that between the solutions of (2.2) and (3.1) when $z_1 \neq 0$ and $z_2 \neq 0$. However, if $z_2 = 0$, the equation for ψ in system (3.1) becomes meaningless. Let us consider the behavior of the solutions as $\rho_2 \to 0$. Let $z_2(0) = 0$; then $\varphi_1(t) \to \varphi_1(0)$ as $t \to 0$. Since $z_2^{\cdot}(0) = B_2 z_1^3$, $\varphi_2(t) \to 3\varphi_1(0) + \psi_2$ or $\varphi_2(t) \to 3\varphi_1(0) + \psi_2 + \pi$. As $z_2(t)$ passes through zero the corresponding $\psi(t)$ jumps by the amount π. If for a solution of (3.1), in which $\rho_2 \to 0$ as $t \to t_0$, a jump by the amount π is prescribed for ψ at $t = t_0$, then the correspondence beteween the solutions of (2.2) and (3.1) is extended to the whole plane $z_2 = 0$. The angular system (3.2) has a singularity at $\theta = 0$. The trajectories of (3.2) when $\theta \neq 0$ are the same as for the system $\theta^{\cdot} = \sin^{1/2}\theta f(\theta, \psi), \psi^{\cdot} = \sin^{1/2}\theta g(\theta, \psi)$. This system has two stationary points $M_1(0, \psi_2)$ and $M_2(0, \psi_2 + \pi)$ on the line $\theta = 0$. These are saddle points when $b_2 \neq 0$. To the solution of (3.1), postulated above, reaching the line $\theta = 0$, corresponds a motion along an incoming separatrix of one of the singular points, an instantaneous jump to the other, and then a motion along an outgoing separatrix of the second point.

4. Necessary stability conditions for the model system. In system (2.2) the plane $z_1 = 0$ is invarient and on it $z_2^{\cdot} = i\omega_2 z_2 + A_{22} z_2 |z_2|^2$. Consequently, the condition $\operatorname{Re} A_{22} = a_{22} < 0$ is necessary for the stability of system (2.2). The other conditions in Criterion A are unnecessary (see Section 8). We now make use of the solutions of system (2.2) of the type of invariant rays.

Lemma 1. For the asymptotic stability of the equilibrium position $z = 0$ of system (2.2) it is necessary that the inequality $\Pi(\theta_*, \psi_*) < 0$ be fulfilled for each steady-state solution (θ_*, ψ_*) of system (3.2).

The proof follows from formula (3.4).

Note 2°. No more than five steady-states (θ_*, ψ_*) exist for system (3.2). Indeed, from the system $f(\theta, \psi) = g(\theta, \psi) = 0$ follows a fifth-degree equation in $\operatorname{tg}\theta$. After the finding of $\operatorname{tg}\theta_*$ the computation of $\Pi(\theta_*, \psi_*)$ requires only algebraic operations; the necessary condition formulated is, in this sense, algebraic. The detailed formulas have been presented earlier(*).

Note 3°. If $\Pi(\theta_*, \psi_*) = \mu > 0$ for some steady-state (θ_*, ψ_*), then from (3.3) we obtain $dR/dt = \mu R^2$ or $d|z|/dt = \mu/2|z|^3$, namely, a growth usual for the solutions of a cubic homogeneous equation (explosive instability /7/). If two such steady-states exist, then, depending on the initial values $\theta(0)$ and $\psi(0)$, we can realize some mode or other of explosive instability.

5. Quadratic Liapunov functions and sufficient stability conditions. Each homogeneous Liapunov function $L_k(z, z^*)$ of system (2.2) is a Liapunov function of system (2.1). This is a consequence of the properties of homogeneous systems /8/ and of the fact in the stability problem system (2.2) is equivalent to a homogeneous system (Sect.2).

Lemma 2. If system (2.2) admists of a quadratic Liapunov function $L_2(z, z^*)$, then:

1°. This same system admits of a Liapunov function of form

$$L_2^\circ = k|z_1|^2 + |z_2|^2 \tag{5.1}$$

2°. System (2.2) remains asymptotically stable under the substitutions $B_1 \to \alpha B_1$ and $B_2 \to \alpha B_2$ $(0 \leqslant \alpha < 1)$. In particular, Criterion A is fulfilled.

Proof. 1°. Let (2.2) admit of the Liapunov function

$$L_2 = k|z_1|^2 + |z_2|^2 + B_{11}z_1^2 + B_{11}^*(z_1^*)^2 + B_{12}z_1z_2 + B_{12}^*z_1^*z_2^* + B_{22}z_2^2 + B_{22}^*(z_2^*)^2 + Cz_1z_2^* + C^*z_1^*z_2$$

If $z_1(t), z_2(t)$ is a solution of system (2.2), then $e^{i\beta}z_1(t), e^{3i\beta}z_2(t)$ is a solution as well (for any real β). Therefore,

$$L_2^\circ = \frac{1}{2\pi}\int_0^{2\pi} L_2(e^{i\beta}z_1, e^{3i\beta}z_2)\,d\beta = k|z_1|^2 + |z_2|^2$$

too is a Liapunov function for system (2.2).

2°. The derivative of L_2° relative to system (2.2) (see (3.1)) is

$$dL_2^\circ/dt = M_k(\rho_1, \rho_2, \psi) = 2\{ka_{11}\rho_1^2 + (ka_{12} + a_{21})\rho_1\rho_2 + a_{22}\rho_2^2 + \rho_1^{3/2}\rho_2^{1/2} \times [kb_1\cos(\psi - \psi_1) + b_2\cos(\psi - \psi_2)]\}$$

By condition, $M_k(\rho_1, \rho_2, \psi) < 0$ for all $\rho_1 > 0, \rho_2 > 0, \psi$. Let

$$m_k(\rho_1, \rho_2) = \max_\psi M_k(\rho_1, \rho_2, \psi) = 2[ka_{11}\rho_1^2 + (ka_{12} + a_{21})\rho_1\rho_2 + a_{22}\rho_2^2 + s^{1/2}\rho_1^{3/2}\rho_2^{1/2}] \tag{5.2}$$
$$s = k^2b_1^2 + b_2^2 + 2kb_1b_2\cos\Delta\psi, \quad \Delta\psi = \psi_2 - \psi_1$$

*) See the footnote**) on p.163.

The function $m_k(\rho_1, \rho_2)$ only decreases under the substitutions $b_1 \to \alpha b_1$ and $b_2 \to \alpha b_2$ $(0 \leqslant \alpha < 1)$ and, therefore, L_2° remains a Liapunov function.

In order to obtain sufficient stability conditions we estimate m_k from above

$$m_k(\rho_1, \rho_2) \leqslant \rho_2^2 P_2(y),\ P_2(y) = c_2 y^2 + c_1 y + c_0, \qquad y = \rho_1 / \rho_2 \tag{5.3}$$
$$c_2 = s^{1/2} + 2ka_{11}, \quad c_1 = s^{1/2} + 2(ka_{12} + a_{21}), \quad c_0 = 2a_{22} < 0$$

If polynomial $P_2(y)$ is negative for all $y > 0$, then $L_2 = k\rho_1 + \rho_2$ is a Liapunov function. The conditions for the negativity of P_2 for $y > 0$ are: 1) $c_2(k) < 0$; 2) $c_1(k) < 0$ or 3) $c_1^2(k) - 4c_2(k)c_0 < 0$. By I_j we denote the collection of positive solutions of inequality j) relative to k.

Theorem 1. Suppose that the c_j have been prescribed by formulas (5.3). For the asymptotic stability of system (2.2) it is sufficient that the interval I_1 intersect either with I_2 or with I_3: $K = I_1 \cap (I_2 \cup I_3) \neq \varnothing$.

Indeed, if $k \in K$, then $L_2 = k\rho_1 + \rho_2$ is a Liapunov function for system (2.2).

Note 4°. Inequalities 1) and 2) reduce to quadratic ones, while 3) to a fourth-degree one. In two important special cases: $\Delta\psi = 0$ and $\Delta\psi = \pi$, inequalities 1) and 2) are linear, while 3) is quadratic.

We fix a_{kl}, satisfying Criterion A, and we consider the asymptotic stability domain Ω in the (b_1, b_2)-plane by virtue of Theorem 1 (the true asymptotic stability domain $\Omega_+ \supset \Omega$). From (5.2) and (5.3) we get that $\Omega(\Delta\psi_1) \subset \Omega(\Delta\psi_2)$ if $\cos(\Delta\psi_1) > \cos(\Delta\psi_2)$. Thus, the case most dangerous for stability is $\Delta\psi = 0$. If $\Delta\psi = \pi$ the stability domain is maximal and can coincide with the whole square $(b_1 \geqslant 0,\ b_2 \geqslant 0)$ when $a_{12} < 0$ and $a_{21} < 0$. Having replaced $\Delta\psi$ by zero, we obtain a structurally-stable sufficient condition (see the footnote on p.163).

Lemma 3. If system (2.2) admits of a Liapunov function in the form of a fourth-degree homogeneous polynomial $L_4(z, z^*)$, then it also admists of a Liapunov function L_4° of form

$$L_4^\circ = k_1 |z_1|^4 + k_2 |z_1|^2 |z_2|^2 + k_3 |z_2|^4 + k_4 [(z_1^*)^3 z_2 + z_1^3 z_2^*]$$

The proof is analogous to that of Lemma 2.

6. Necessary stability conditions for the complete system.

Theorem 2. Let the angular system (3.2) have a structurally-stable steadystate $P_* = (\theta_*, \psi_*)$, $0 < \theta_* < \pi/2$, and let $\Pi(\theta_*, \psi_*) = \Pi_* > 0$. Then the steadystate $u = 0$ of system (1.1) is unstable.

Proof. Let $M = \| (d(f, g) / d(\theta, \psi))_{\theta=\theta_*, \psi=\psi_*} \|$ and let μ_1 and μ_2 be eigenvalues of matrix M. By hypothesis, $\operatorname{Re}\mu_k \neq 0$. Three cases are possible: a) $\operatorname{Re}\mu_1 < 0$, $\operatorname{Re}\mu_2 < 0$; b) $\operatorname{Re}\mu_1 > 0$, $\operatorname{Re}\mu_2 > 0$; c) $\mu_1 < 0$, $\mu_2 < 0$. We consider them in turn. We normalize system (1.1) up to third-order terms, inclusive, and in (2.1) we introduce the variables R, θ, ψ, τ (see Sect.3).
We obtain

$$\frac{dR}{d\tau} = R[\Pi(\theta, \psi) + \delta_0], \quad \frac{d\theta}{d\tau} = f(\theta, \psi) + \delta_1, \qquad d\psi/d\tau = g(\theta, \psi) + \delta_2 \tag{6.1}$$

Here $|\delta_k(R, \theta, \varphi_1, \varphi_2)| \leqslant CR$ for $0 < \theta_1 \leqslant \theta \leqslant \theta_2 < \pi/2$, $k = 0, 1$. Let U_0 be a neighborhood of P_*, in which $dR/d\tau > {}^1/_2\,\Pi_* R$ when $R < R_0$.

a) Let $l(\theta, \psi)$ be a quadratic Liapunov function for the steadystate P_* of system (3.2) and let $dl/d\tau \leqslant -ql$ relative to system (3.2) in some neighborhood $U_1(P_*) \subset U_0$. For $l_* > 0$ let the line $l(\theta, \psi) = l_*$ lie wholly in U_1. In a four-dimensional space R^4 we specify the domain Ω by the conditions $R < R_0$, $0 \leqslant l(\theta, \psi) \leqslant l_*$. In domain Ω, $dR/d\tau > {}^1/_2\,\Pi_* R$. On the boundary $l = l_*$ we have, relative to system (2.1), $dl/d\tau < -ql_* + CR < -{}^1/_2 ql_*$ when $R < R_1$ (we can take $R_1 < R_0$). If $R(0) = \varepsilon < R_1$ and $l(\theta(0), \psi(0)) < l_*$, then as τ grows the function $R(\tau)$ increases monotonically up to the value R_1. The instability has been proved because ε is arbitrary.

b) Let $l(\theta, \psi)$ be a Liapunov function for the steadystate P_* of system (3.2) as $\tau \to -\infty$. We select Ω in the same way as in a). Relative to (2.1) let $dl/d\tau > {}^1/_2 ql_*$ on $l = l_*$ for $R < R_2 < R_0$. If $R(0) = R_2$ and $l(\theta(0), \psi(0)) \leqslant l_*$, then as $\tau \to -\infty$ the function $R(\tau)$ monotonically tends to zero, i.e., the trajectory tends to the steadystate $z = 0$. The presence of one such trajectory is sufficient for instability as $\tau \to \infty$.

c) We reduce the original system $du/d\tau = f(u)$ to normal form up to terms of order $2n + 3$, inclusive, $n = [\mu_2/\Pi_*] + 1$ ($n = \mu_2/\Pi_*$ when μ_2/Π_* is an integer). We introduce the varibles R, θ, ψ, τ and by β_1 and β_2 we denote two linear combinations of $\theta - \theta_*$ and $\psi - \psi_*$ diagonalizing matrix M. We introduce a scale on τ such that $\Pi_* = 1$. We obtain

$$R^{\cdot} = R(1 + \delta_0) \quad (^{\cdot} \equiv d/d\tau) \tag{6.2}$$

$$\beta_1^{\cdot} = \mu_1\beta_1 + \sum_1^n c_k^{(1)} R^k + \delta_1 \qquad \beta_2^{\cdot} = \mu_2\beta_2 + \sum_1^{n-1} c_k^{(2)} R^k + c_n^{(2)} R^n + \delta_2 \tag{6.3}$$

Here $\delta_j(R, \beta, \varphi)$ are smooth functions when $|\beta| < \varepsilon_0$, and

$$|\delta_0| \leqslant C_4 |\beta| + C_5 R, \quad |\delta_k| \leqslant C_1 R |\beta| + C_2 |\beta|^2 + C_3 R^{n+1}, \quad k = 1, 2; \quad |\beta|^2 = \beta_1^2 + \beta_2^2$$

We set

$$\beta_1 = \alpha_1 + \sum_1^n s_k R^k, \quad \beta_2 = \alpha_2 + \sum_1^{n-1} \sigma_k R^k \tag{6.4}$$

We choose s_k and σ_k so that the sum in (6.3) vanishes. We obtain

$$R^{\cdot} = R(1 + \Delta_0), \quad |\Delta_0| \leqslant CR \tag{6.5}$$

$$\alpha_1^{\cdot} = \mu_1 \alpha_1 + \Delta_1, \quad \alpha_2^{\cdot} = \mu_2 \alpha_2 + c_n^{(2)} R^n + \Delta_2 \tag{6.6}$$

$$|\Delta_k| \leqslant C_1 |\alpha|^2 + C_2 R |\alpha| + C_3 R^{n+1}, \quad k = 1, 2; \quad |\alpha|^2 = \alpha_1^2 + \alpha_2^2$$

In the general position case, when μ_2 is not an integer, we exclude as well the term $c_n^{(2)} R^n$ in the second equation in (6.6). Let $\Phi(\tau)$ be a solution of the equation

$$\Phi^{\cdot} = \mu_2 \Phi + c_n^{(2)} e^{n\tau}, \quad \Phi(0) = 0, \qquad \text{(for nonintegral } \mu_2 \ \Phi \equiv 0).$$

We set $F = \alpha_1^2 + (\alpha_2 - \Phi(\ln R))^2 - R^{2n+1}$ and $\Omega = \{\alpha, R : F(\alpha_1, \alpha_2, R) < 0, R < R_1\}$. In domain Ω $|\alpha_1| \leqslant R^{n+1/2}$ and $|\alpha_2 - \Phi(\ln R)| \leqslant R^{n+1/2}$. We treat Ω as a domain in the original four-dimensional space R_u^4. Computing the derivative of F relative to (1.1), in domain Ω we obtain

$$F^{\cdot} < -R^{2n+1}(1 + C |\ln R| R^{1/2})$$

by using (6.5)—(6.6). Hence, F is a Chetaev function and the proof has been completed.

Notes. 5°. The constructions used in parts a) and b) of the proof go back to Chetaev /9/.

6°. The structural stability of P_* has been assumed only to shorten the proof. Here, for a large magnitude of μ_2 / Π_* a high smoothness is assumed for the right hand sides of system (1.1). This requirement is not central to the matter at hand and is the price we pay for the proof presented.

7°. Theorem 2, in the formulation presented, is valid for several angular variables and in this form is applicable to a number of cases, including the one considered in /6/.

Theorem 3. For system (2.1) let $a_{22} > 0$ and $a_{22} - a_{12} \neq 0$. Then the steadystate $z = 0$ is unstable.

Proof. We choose $\delta_0 > 0$ such that $\Pi(\theta, \psi) > a_{22}$ when $\theta > (\pi/2) - \delta_0$, $(\Pi(\pi/2, \psi) = 2a_{22}$; see (3.3)). Relative to (2.1) let $dR/d\tau > {}^1/_2\, a_{22} R$ when $\theta > (\pi/2) - \delta_0$ and $R < R_0$. Let $a_{22} - a_{12} = \mu < 0$. Then the inequality $|f(\theta, \psi)| > |\mu|((\pi/2) - \theta)$ is valid in domain $\theta > (\pi/2) - \delta_1$ for a sufficiently small $\delta_1 < \delta_0$. Let $\Omega = \{R, \theta, \psi: \theta > (\pi/2) - \delta_1, R < R_1\}$. We convince ourselves of the instability by repeating the arguments of Theorem 2 (case b)). When $\mu > 0$ the proof is analogous to case a) of Theorem 2.

7. Limit cases of small and large resonance coefficients.

Theorem 4. Let system (2.3) be asymptotically stable. Then $\varepsilon > 0$ exists such that system (2.1) is asymptotically stable when $b_1 < \varepsilon$ and $b_2 < \varepsilon$.

Proof. The Liapunov function $L = k\rho_1 + \rho_2$ for system (2.3) is, by virtue of (2.4), a Liapunov function for (2.1) when ε is sufficiently small.

Theorem 5. Let system (2.3) be structurally-stably unstable. Then $\varepsilon > 0$ exists such that system (2.1) is unstable when $b_1 < \varepsilon$ and $b_2 < \varepsilon$.

Proof. Three cases are possible.

1°. Condition $a_{11} > 0$ is fulfilled. Then $\Pi_1(\theta) > a_{11}$ when $\theta \leqslant \theta_0$ (see (3.3)). When $b_k < {}^1/_8 a_{11}$ $(k = 1, 2)$ we have $\Pi(\theta, \psi) > {}^1/_2\, a_{11}$ for $\theta \leqslant \theta_0$. We choose $\delta < \theta_0$ such that $\mu = f_1(\delta) \neq 0$. Then $\operatorname{sign} f(\delta, \psi) = \operatorname{sign} \mu$ when $|b_k| < {}^1/_8 |\mu|$. We set $\varepsilon = {}^1/_8 \min(a_{11}, |\mu|)$ and $\Omega = \{\theta \leqslant \delta, R < R_1\}$. The subsequent arguments are analogous to the proof of Theorems 2 and 3.

2°. Condition $a_{22} > 0$ is fulfilled. The proof is analogous to case 1°; the domain Ω is specified by the inequalities $\theta \geqslant (\pi/2) - \delta$, $R \leqslant R_2$.

3°. $a_{11} < 0$, $a_{22} < 0$, $a_{12} > 0$, $a_{21} > 0$, $a_{11}a_{22} - a_{12}a_{21} < 0$. In this case system (2.3) has a solution of form (2.5) with $p \neq 0$. For system (3.2) with $b_1 = b_2 = 0$ this signifies the presence of an attracting limit cycle $\theta = \theta_*$: $f_1(\theta_*) = 0$, $f_1'(\theta_*) < 0$, $\Pi_1(\theta_*) = \Pi_* > 0$. Let $\delta > 0$ be such that $\Pi_1(\theta) > {}^1/_2 \Pi_*$, $f(\theta_* - \delta) < 0$ and $f(\theta_* + \delta) < 0$ when $|\theta - \theta_*| < \delta$. Having set $\Omega = \{|\theta - \theta_*| < \delta, R < R_1\}$, we obtain instability, as in part a) of Theorem 2.

Lemma 4. Let $b_1 > 0$, $b_2 > 0$, $\Delta\psi = \psi_1 - \psi_2 = \pi$. Then the system $f_2(\theta, \psi) = g_2(\theta, \psi) = 0$ (see (3.2)) has a solution (θ_*, ψ_*) such that: 1) matrix $d(f_2, g_2)/d(\theta, \psi)$ has no eigenvalues on the imaginary axis when $\theta = \theta_*$ and $\psi = \psi_*$; 2) $\Pi_2(\theta_*, \psi_*) > 0$.

The lemma's proof is omitted.

Theorem 6. In system (2.1) let the coefficients A_{kl} and the value $\Delta\psi \neq \pi$ be fixed. Let $\beta_1 > 0$, $\beta_2 > 0$, $b_1 = \beta_1 / \varepsilon$, $b_2 = \beta_2 / \varepsilon$. Then $\varepsilon_0 > 0$ exists such that system (2.1) is unstable for $0 < \varepsilon < \varepsilon_0$.

Proof. In (3.2) and (3.3) we introduce the change $\tau = \varepsilon\tau_1$. We obtain

$$d\theta / d\tau_1 = f_\varepsilon,\ d\psi / d\tau_1 = g_\varepsilon,\ dR / d\tau_1 = R\Pi_\varepsilon \tag{7.1}$$
$$f_\varepsilon = f_2 + \varepsilon f_1,\quad g_\varepsilon = g_2 + \varepsilon g_1,\quad \Pi_\varepsilon = \Pi_2 + \varepsilon\Pi_1$$

Here β_k replaces b_k in f_2, g_2, Π_2. For a sufficiently small ε, by virtue of Lemma 4 the system $f_\varepsilon = g_\varepsilon = 0$ has a solution (θ_*, ψ_*) for which $\Pi_\varepsilon(\theta_*, \psi_*) > 0$ and matrix $d(f_\varepsilon, g_\varepsilon) / d(\theta, \psi)$ has no eigenvalues on the imaginary axis. Applying Theorem 2 to (7.1), we get that system (7.1) and, by the same token, system (2.1) are unstable.

Note 8°. The theorem's assertion is invalid when $\Delta\psi = \pi$: system (2.1) can be asymptotically stable for any $b_1 \geqslant 0$ and $b_2 \geqslant 0$. For an arbitrary $\Delta\psi \neq \pi$ the large magnitude of only one of the resonance coefficients $(b_1$ or $b_2)$ does not guarantee instability.

8. Example of asymptotic stability when the conditions of Criterion A are not fulfilled: $a_{11} > 0$.

We consider a system of type (2.2)

$$z_1^{\cdot} = i\omega z_1 + z_1(|z_1|^2 - 12|z_2|^2), \qquad z_2^{\cdot} = 3i\omega z_2 + z_2(-14|z_1|^2 - 4|z_2|^2) - 18z_1^3$$
$$(a_{11} = 1,\ a_{21} < 0,\ a_{12} < 0,\ a_{22} < 0,\ B_1 = 0,\ B_2 \neq 0,\ \psi_2 = \pi)$$

The Liapunov function for this system is a fourth-degree homogeneous polynomial (compare with Lemma 3)

$$L_4 = {}^3/_2 |z_1|^4 + |z_2|^4 + {}^1/_2 (z_1^3 z_2^* + (z_1^*)^3 z_2)$$

Here it is essential that $B_2 \neq 0$; when $B_2 = 0$ the inequality $a_{11} < 0$ is necessary for stability. We remark that by virtue of Lemma 2 a Liapunov function in the form of a quadratic polynomial does not exist when $a_{11} > 0$.

9. Concluding notes. On sufficient stability conditions.

Sufficient stability conditions can be obtained from the requirement that one of the homogeneous polynomials $L_k(z, z^*)$ of fixed degree k is a Liapunov function. For problems of asymptotic stability of a steadystate, in which algebraic stability criteria have been found, the stability is usually ensured by a quadratic Liapunov function $L_2(z, z^*)$ $(k = 2)$. In the problem at hand effectively verifiable algebraic conditions exist that are necessary and sufficient for system (2.1) to admit of a quadratic Liapunov function. These conditions, obtained from the Sturm criterion, are cumbersome and are not cited here, and this is justified by the fact that $L_2(z, z^*)$ is not suitable in the cases of most interest. The use of L_k with $k > 2$ is sometimes useful (see Sect.8), but from the nonexistence of an algebraic criterion it follows that asymptotically stable systems (2.1) not admitting of a Liapunov function L_k with $k \leqslant k_0$ exist for an arbitrarily large k_0.

On necessary stability conditions. In order to evaluate the significance of the necessary conditions presented above, we note that if the angular system (3.2) has a limit cycle $\gamma = (\theta(s), \psi(s))$, then for stability it is necessary that

$$I_\gamma = \oint \Pi(\theta(s), \psi(s))\, ds < 0$$

The signs of these integrals are not taken into account by the necessary conditions presented in the paper. When the signs of the integrals over the limit cycles are taken into account, the necessary conditions presented above become sufficient as well if the angular system is structurally stable.

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REFERENCES

1. KAMENKOV, G. V., On stability of motion. Sb. Tr. Kazansk. Aviats. Inst., No.9, 1939.

2. KUNITSYN, A. L., On stability in the critical case of pure imaginary roots with internal resonance. Differents. Uravn., Vol.7, No.9, 1971.

3. KHAZINA, G. G., Certain stability questions in the presence of resonance. PMM Vol.38, No.1, 1974.

4. KHAZIN, L. G. and SHNOL', E. E., Elementary cases of algebraic unsolvability in asymptotic stability problems. Dokl. Akad. Nauk SSSR, Vol.240, No.6, 1978.

5. PLISS, V. A., The reduction principle in stability theory. Izv. Akad. Nauk SSSR, Ser. Mat., Vol.28, No.6, 1964.

6. MOLCHANOV, A. M., Stability in the case of neutrality of the linear approximation. Dokl. Akad. Nauk SSSR, Vol.141, No.1. 1961.

7. BARENBLATT, G. I. (Ed.), Nonlinear Theory of Wave Propagation. Moscow, "Mir", 1970.

8. KRASOVSKII, N. N., On stability in the first approximation. PMM Vol.19, NO.5, 1955.

9. CHETAEV, N. G., Papers on Analytical Mechanics. Moscow, Izd. Akad. Nauk SSSR, 1962.

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